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Bed-Load Dispersion: A Literature Review

James W. Lewis, Alejandro Sánchez, Stanford Gibson,
Travis Dahl, and Ian Floyd

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Bed-Load Dispersion: A Literature Review

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Abstract

This Special Report documents available research pertaining to the dispersion of bed-load sediment transport, specifically the dispersion coefficient. This knowledge can inform the evaluation and improvement of existing U.S. Army Corps of Engineers (USACE) sediment transport models. A discussion is presented on normal and anomalous dispersion. Several aspects of bed-load dispersion are discussed, including whether the assumption of normal dispersion and isotropy are valid. Numerical dispersion coefficients are relative to the context of all considerations within a modeling framework and affect sediment erosion, deposition, and the resulting water quality. As witnessed by the number of recent publications, sediment bed-load dispersion is an active area of research, and the published values for the bed-load dispersion coefficient vary greatly.

This document is intended for scientists and engineers with a basic understanding of physics and numerical modeling of sediment transport.

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Preface

This study was conducted by the Coastal Inlets Research Program (CIRP), which is funded by the Operation and Maintenance (O&M) Navigation business line of the Headquarters, U.S. Army Corps of Engineers (HQUSACE) under Project#405370, “Bed-Load Sediment Transport Research.” The CIRP is administered for Headquarters by the U.S. Army Engineer Research and Development Center (ERDC), Coastal and Hydraulics Laboratory (CHL), Vicksburg, MS, under the Navigation Research, Development, and Technology (RD&T) Portfolio. At the time of this study, Jeffrey A. McKee, Chief, Navigation Branch, HQ was the HQUSACE Navigation Business Line Manager, W. Jeff Lillycrop, CHL, was the ERDC Technical Director for Civil Works and Navigation RD&T, Charles E. Wiggins was the ERDC Associate Technical Director for Navigation, and Dr. Julie Rosati, CHL, was the CIRP Program Manager.

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The work discussed herein was performed by the River Engineering Branch (CEERD-HF-R) of the Flood and Storm Protection Division (CEERD-HF), ERDC-CHL; and the Institute for Water Resources, Hydrologic Engineering Center (IWR-HEC), Davis, CA. At the time of publication, Keith Flowers was Chief, (CEERD-HFR); and Ty Wamsley was Division Chief, (CEERD-HF). The Deputy Director of ERDC-CHL was Dr. Kevin Barry and the Director was José E. Sánchez.

The Commander of ERDC was COL Bryan S. Green, and the Director was Dr. Jeffery P. Holland.

Unit Conversion Factors

Multiply	By	To Obtain
feet	0.3048	meters
inches	0.0254	meters
square feet	0.09290304	square meters
square inches	6.4516 E-04	square meters

1 Introduction

Research concerning coastal and riverine sediment transport, specifically erosion and deposition, affects public safety, water resource management, and operations and maintenance practices. USACE developed and successfully applied several numerical morphologic and sediment particle tracking models (e.g., Tate et al. 2009; Sánchez and Wu 2011; Gibson et al. 2010); however, existing representation of sediment transport is limited and numerical calculations often vary by more than an order of magnitude. As part of the continuing model development, the present work focuses on improving the dispersion coefficient and dispersion equations used for bed-load sediment transport. For the discussion herein, sediment advection can be understood as the transport of sediment along with the general flow of the water, which is typically related to the average velocity. Sediment dispersion describes how the sediment particles spread relative to one another as they are being transported. Bed-load dispersion concerns the spreading of sediment particles that are moving along the bed of the water body.

Accurate simulation of the bed-load dispersion is important for understanding and predicting the spread of specific materials. Applications where users are interested in tracking the transport, mixing, and sedimentation of specific materials, perhaps a specific grain size of sediment or a contaminant, would benefit from improved dispersion modeling capabilities. For these applications, greater accuracies in estimating the arrival and distribution of material concentrations are important.

Historic sediment transport studies have focused on estimating the equilibrium sediment transport rate (e.g., Yalin 1977; Meyer-Peter and Müller 1948) based on solving the continuity, or Exner, equation (Parker et al. 2000) over a control volume. Methods, such as Meyer-Peter and Müller, computed mass change, but could not capture the movement of particles or pulses explicitly. More recently, several studies have focused on sediment wave scour rates (Abraham et al. 2011) and particle velocity (Lajeunesse et al. 2010; Furbish et al. 2012b), which are the primary components of sediment advective transport. However, there have been relatively few studies of sediment dispersion (e.g., Einstein 1972; Hubbell and Sayer,

1964; Drake et al. 1988) before the last decade, especially bed-load dispersion. Furthermore, a general lack of consensus on sediment dispersion within the literature exhibits a need for further research.

1.1 Approach

Recently, an ERDC-CHL/IWR-HEC team of Gibson et al. (2016) conducted laboratory experiments of bed-load sediment transport to measure dispersion. They tracked colored sediment pulses along a flume under steady equilibrium sediment transport. Bed samples were analyzed to study bed-load advection and dispersion. The experiments serve as a valuable validation dataset for sediment transport models and provide insight into the physical processes of bed-load transport. Two USACE sediment transport models, namely the Adaptive Hydraulics (AdH) model with the sediment library (SEDLIB) and the Hydrologic Engineering Center's River Analysis System (HEC-RAS), will be tested with results from Gibson et al. (2016) and the relationships discussed herein, which will inform any recommended changes to the models.

AdH hydrodynamic model with SEDLIB (Brown et al. 2014) was developed at the Engineer Research and Development Center's (ERDC) Coastal and Hydraulics Laboratory (CHL). In addition, the Adh/SEDLIB is undergoing further development, and it is available to the public. AdH is capable of simulating 2-D or 3-D hydraulics along with sediment modeling and the transport of conservative constituents, such as dye clouds. The unique aspect of AdH is its ability to 'adapt' the model domain mesh in certain locations or at certain times where more resolution may be needed. Sediment transport calculations are performed within the SEDLIB of AdH. For each step of the user-specified time increment, AdH resolves the hydrodynamics (i.e., velocities, water depths, etc.) first, and then passes that information to the sediment module. The sediment time-step is typically smaller than the hydrodynamic time-step, i.e., multiple sediment time-steps are taken within each hydrodynamic time-step. Within the sediment module, the suspended sediment concentration and bed-load transport are iteratively solved until they converge to a change in displacement, which is less than the smallest grain diameter. Once converged, the water depth is adjusted based on the converged displacement in order to keep a constant water surface elevation, and the water velocity is adjusted in order to keep a constant flow rate. The sediment routine is repeated until it reaches the hydrodynamic time step, and then the entire procedure is repeated, starting with the next hydrodynamic time-step.

HEC-RAS is a widely applied one- and two-dimensional (2D) unsteady flow model available to the public (U.S. Army Corps of Engineers 2016). HEC-RAS calculates unsteady open channel hydraulics using user-defined cross-sections. For each cross-section, the one-dimensional framework uses depth-averaged and width-averaged parameters. For each time-step, HEC-RAS uses the sediment continuity equation and the sediment transport capacity to determine the change in sediment volume. Bed elevation change and grain size distributions are subsequently calculated at all nodes of each cross-section.

1.2 Objective

The objective of this literature review is to prepare the conceptual foundation for the continued analysis of the flume experiment data and future numerical modeling improvements related to bed-load dispersion. This report provides a background summary of dispersion related to solute transport, because bed-load dispersion is influenced by the same processes as solute dispersion. However, bed-load dispersion is also caused by additional factors such as bed mixing, bed forms, etc. which are discussed in more detail in Section 3.

The report is organized as follows:

- Section 2 explains the background of diffusion and dispersion concepts.
- Section 3 is focused on literature specifically related to bed-load dispersion.
 - Section 3.1 summarizes the available literature on longitudinal dispersion using normal (Section 3.1.1), anomalous (Section 3.1.2), and fractional advection-dispersion (AD) relationships.
 - Section 3.2 provides a summary of transverse dispersion literature.

2 Background

When the applied skin friction bed shear stress exceeds a critical threshold, non-cohesive sediment particles will begin to move. Observations of the transition show long streaks of particles in motion very close to the bed with occasional bursts into the water column. Some particles will be suspended into the water column (suspended load) and others will roll, jump, and/or slide along the bed (bed-load).

Sediment transport has been studied thoroughly since the seminal work of Paul Francois DuBoys in 1879. Early studies of sediment transport focused on predicting the equilibrium sediment transport rate and computing the bed change by solving a continuity equation (e.g., Meyer-Peter and Müller, 1948; Bagnold, 1956). These formulations computed mass change, but did not explicitly account for the stochastic behavior of particles. Recent research suggests that the non-equilibrium and stochastic behavior of bed-load transport can be predicted using AD models.

2.1 Diffusion and dispersion concepts

Diffusion is the process by which a constituent ‘spreads’ due to molecular movements, collisions, and turbulent fluctuations. Brown (1828) observed the apparently random movement of pollen particles suspended in water. The phrase ‘random walk’, first introduced by Karl Pearson (1905), is sometimes used synonymously with Brownian motion, where a particle’s path consists of step angles and distances, which are completely independent from previous steps. Albert Einstein was the first to successfully explain the cause of Brown’s observations due to the stochastic behavior of the surrounding fluid molecules (Einstein 1905). Fick (1855) identified that in the process of solute diffusion within a fluid, the flux is proportional to the concentration gradient of the solute, which became known as Fick’s first law of diffusion (in one-dimension):

$$q = -D \frac{\partial C}{\partial x} \quad (1)$$

where:

q = diffusive mass flux [$M T^{-1} L^{-1}$]

C = concentration [M/L^3]

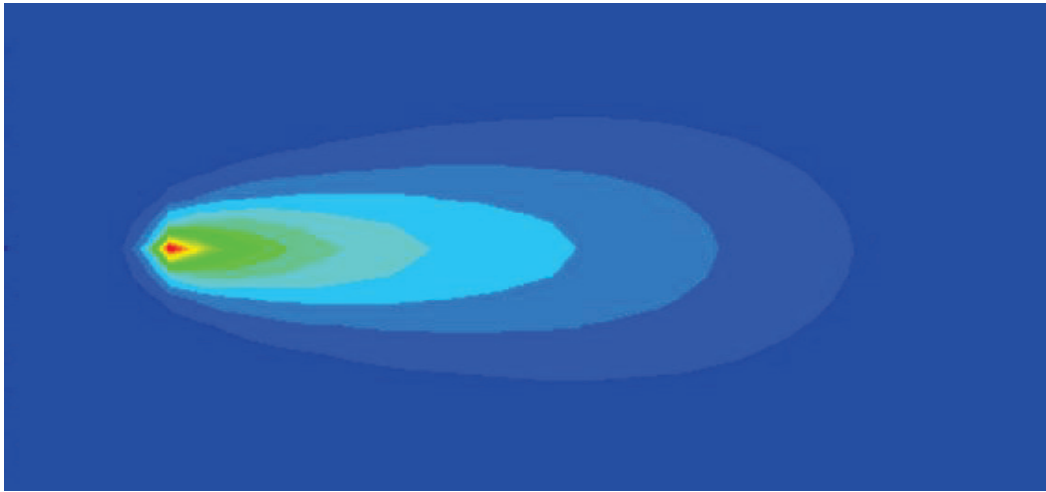
D = diffusion coefficient [L^2/T]

‘Dispersion’ also describes constituent ‘spreading’ processes. These words, diffusion and dispersion, are often used interchangeably or, at least, with overlapping semantic range. However, dispersion usually refers to macro-scale processes, which occur due to the non-uniform movements of the surrounding fluid. Holley (1969) clarified the terminology, defining ‘dispersion’ as a deviation from the temporal average convection velocity, and diffusion as the “difference between the true convection in a given direction and the spatial average of convection in that direction.” Any other localized changes to the flow, such as around structures or geometric features of cross-sections, can also impact the dispersion rate. Although dispersion can behave very similar to diffusion, dispersive spreading is generally one or two orders of magnitude greater than molecular diffusion. Additionally, in sediment transport, where the ‘discrete packets’ of transportable mass (sediment particles) are larger than the molecular scale, typically only macro-scale processes are applicable.

Fick’s law of diffusion may accurately describe many scenarios of dispersion, but it is generally an extrapolation of the original derivation based on the molecule-scale.

Figure 1 visually shows a simple numerical representation of how a continuous point source could theoretically spread within a 2-D rectangular channel having a uniform velocity from left to right.

Figure 1. Example of 2-Dimensional advection-dispersion
(advection from left to right).



2.2 Dispersion coefficient

Dispersion coefficients (K) are difficult to measure or estimate. Sir Geoffrey Taylor was responsible for most of the early literature on these coefficients, including:

- Dispersion by continuous movements (Taylor 1921),
- Dispersion in laminar flow through a tube (Taylor 1953), and
- Dispersion in turbulent flow through a pipe (Taylor 1954).

In general, Taylor's work explained how dispersion occurs via the velocity variations within the fluid flow. Elder (1959) applied Taylor's analysis to solute dispersion within open-channels assuming a von Karman logarithmic velocity profile in order to obtain a theoretical dispersion coefficient using basic properties of the flow:

$$K = 5.93u_*h \quad (2)$$

where:

- h = water depth [L]
- u_* = friction (or shear) velocity [L/T]; $u_* = \sqrt{ghS}$
- g = gravitational acceleration [L/T²]
- S = channel slope [dimensionless]

Fischer et al. (1979) demonstrated that dispersion coefficients vary widely in real streams, but are usually much greater than Elder's result. For real streams, Fischer et al. (1979) developed the following relationship for the dispersion coefficient:

$$K = 0.011 \frac{U^2 W^2}{u_* h} \quad (3)$$

where:

U = mean depth-averaged velocity [L/T]

W = channel width [L]

Equation 3 agreed with most study observations within approximately a factor of four. Fisher et al. (1979) noted that there is uncertainty in determining a specific dispersion coefficient in real streams since every irregularity contributes to dispersion, but the majority of streams are uniform enough for an approximate analysis.

Many methods of estimating the solute transport dispersion coefficient have similar forms (McQuivey and Keefer 1974; Liu 1977; Iwasa and Aya 1991; Koussis and Rodriguez-Mirasol 1998; Seo and Cheong 1998; Deng et al. 2002; Kashefipour and Falconer 2002; and Sahay and Dutta 2009), with different coefficients, as shown in Table 1. Disley et al. (2015) compared the equations listed in Table 1 in more depth and showed an improved behavior over a large number of dye tracing field tests with the 2015 equation as listed in Table 1.

Table 1. Solute transport dispersion coefficient equations found in the literature.

Investigator	Empirical Equation (*)
Disley et al. (2015)	$\frac{K}{u_* h} = 3.563 \left(\frac{\bar{u}}{\sqrt{gh}} \right)^{-0.4117} \left(\frac{W}{h} \right)^{0.6776} \left(\frac{\bar{u}}{u_*} \right)^{1.0132}$
McQuivey and Keefer (1974)	$\frac{K}{u_* h} = 0.058 \left(\frac{1}{S} \right) \left(\frac{\bar{u}}{u_*} \right)$
Liu (1977)	$\frac{K}{u_* h} = 0.18 \left(\frac{W}{h} \right)^2 \left(\frac{\bar{u}}{u_*} \right)^{0.5}$
Iwasa and Aya (1991)	$\frac{K}{u_* h} = 2 \left(\frac{W}{h} \right)^{1.5}$

Investigator	Empirical Equation (*)
Koussis and Rodriguez-Mirasol (1998)	$\frac{K}{u_* h} = 0.6 \left(\frac{W}{h} \right)^2$
Seo and Cheong (1998)	$\frac{K}{u_* h} = 5.195 \left(\frac{W}{h} \right)^{0.62} \left(\frac{\bar{u}}{u_*} \right)^{1.428}$
Deng et al. (2002)	$\frac{K}{u_* h} = 0.15 \left(\frac{1}{8\epsilon} \right) \left(\frac{W}{h} \right)^{1.667} \left(\frac{\bar{u}}{u_*} \right)^{0.5}$ $\epsilon = 0.145 + \left(\frac{1}{3520} \right) \left(\frac{W}{h} \right)^{1.38} \left(\frac{\bar{u}}{u_*} \right)$
Kashefipour and Falconer (2002)	$\frac{K}{u_* h} = 10.612 \left(\frac{\bar{u}}{u_*} \right)^2$
Sahay and Dutta (2009)	$\frac{K}{u_* h} = 2 \left(\frac{W}{h} \right)^{0.96} \left(\frac{\bar{u}}{u_*} \right)^{1.25}$

*Note: See Appendix for symbol definitions

The purpose of presenting the solute dispersion coefficient equations in Table 1 is to compare the forms of the equations that could be useful for understanding bed-load dispersion. Bed-load dispersion is influenced by the same factors as solute dispersion, but it is also influenced by a number of other complexities specific to bed-load sediment transport, as discussed in the next section.

3 Bed-Load Dispersion

The primary focus of this report is the dispersion of sediment particles that are being transported along the bed. In general, bed-load dispersion is caused by spatial and temporal variations in particle velocities. The variations in particle velocities can be caused by many processes, including variations in the advective and turbulent fluid velocities, particle collisions, sediment mixing in the bed, armoring, bedforms, sediment clusters, flow belts, channel morphology, and variations in particle characteristics such as shape, density, and size. Additionally, natural environments have many other causes of bed-load dispersion: vegetation, structures, fish, insects, and macroinvertebrates. All of the above processes cause bed sediments to transport non-uniformly and hence produce dispersion. Haschenburger (2013) offers insight about many causative factors, such as those mentioned above, through a 17-year field study of tagged gravels in the Carnation Creek.

Different modeling frameworks may choose to simulate processes mentioned in the previous paragraph independently from their dispersive bed-load transport. For example, simulating bed mixing, sorting, or armoring separately can cause bed-load dispersion within the model on its own, but there is also bed-load dispersion caused by many processes that may not be modeled, such as listed in the previous paragraph. A representative dispersion coefficient that works within one modeling framework may not be comparable to a dispersion coefficient within a model that handles the processes differently. For example, the K value for bed-load dispersion in a model that simulates bed mixing as a separate process is expected to be different from the K value in a model that does not simulate bed mixing. Along these lines, the work of Pelosi et al. (2016) studied two different numerical modeling frameworks of how to handle vertical mixing and demonstrated that the second Exner Based Master Equation (2D EBME) (Pelosi et al. 2014), which explicitly included vertical movement, represented field data better than the active layer formulation EBME-A. In general, a numerical dispersion coefficient may vary significantly between models and is relative to the specific context of each model's considerations.

3.1 Longitudinal dispersion

3.1.1 Normal dispersion

The literature diverges between those who consider the bed-load dispersion as normal (Fickian) behavior, and those who consider it anomalous (non-Fickian). This section will discuss research on normal dispersion while the next section will discuss anomalous dispersion.

Under normal dispersive behavior, a Gaussian distribution typically represents particles spreading, and the Fickian analogy from diffusion can be expanded to dispersion of saltating grains existing in three states: motion, stationary, and buried. The observed mean and variance of particle displacements from experimental measurements at specific time snap-shots can be used to determine a dispersion coefficient. The variance of particle displacements, $V(x)$, can be calculated by:

$$V(x) = \overline{(x - \bar{x})^2} = \overline{x^2} - \bar{x}^2 \quad (4)$$

where:

- $\bar{\quad}$ = notation for representing a mean value, averaged spatially
- \bar{x} = mean particle displacement, averaged over the individual particles
- $\overline{x^2}$ = mean of the squared particle displacements, averaged over the individual particles

$V(x)$ is also referred to as the mean squared displacement (MSD). Einstein (1905) showed that the variance of the particle displacements can be linearly related to time using the dispersion coefficient when taking a limit as the number of particle steps approaches infinity. Following that reasoning and using the assumption of normal Fickian dispersion, the equation (Taylor 1921) for relating variance of particle displacements to the dispersion coefficient, K , is:

$$V = 2Kt \quad (5)$$

Several descriptive studies measured bed-load tracer dispersion, in the lab (Wong et al, 2007; Gibson, 2009; Gibson et al., 2011) and the field (Kennedy and Kouba, 1970; Ferguson et al, 2002; Carling et al., 2006;

Haschenburger 2013; Hassan, 2013), but did not collect data that could be translated into coefficients.

Hubbell and Sayre (1964) (Sayre and Hubbell 1965) performed the first bed-load dispersion experiments, using radioactive tracers to track sand transport over the course of several days. As radioactive methods fell out of favor, however, subsequent studies focused on coarser gravel particles that are easier to distinguish visually.

Drake et al. (1988) introduced 125 orange-painted 4–8 mm particles onto the bed of Duck Creek near Pinedale, Wyoming, photographing their positions every 15 sec for 240 sec. Since the particles appeared to disperse more quickly in the first part of the experiment, dispersion coefficients were estimated over the final 180 sec of the experiment. Assuming normal behavior, Drake et al. computed dispersion coefficients of approximately 4.6 cm²/s in the longitudinal direction and 0.26 cm²/s in the transverse direction. However, the caption of their plot (Drake et al. 1988, and Figure 9b) raises a concern about the accuracy of their estimates, because they state that their dispersion rates were based on fitting straight lines “by eye” between $\sqrt{V(x)}$ and t , instead of between $V(x)$ and t . It is possible that the estimated values of 4.6 and 0.26 cm²/s were calculated correctly, and that it is just their figure caption which is misleading.

Chang and Yen (2002) assumed bed-load sediment transport can be modeled with the classic advection-dispersion equation (ADE) in one-dimension:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = K \frac{\partial^2 C}{\partial x^2} \quad (6)$$

where u = mean bed-load material velocity [L/T], K = bed-load dispersion coefficient [L²/T], and C is the bed-load material concentration with units usually expressed as mass per unit area or volume per unit area. Chang and Yen (2002) estimated the dispersion coefficient through dimensional analysis, which included the following dimensionless quantities:

$$\frac{K}{u_* d} = f \left(S_{oe}, m, \frac{v^2}{(s-1)gd^3}, \frac{d}{h} \right) \quad (7)$$

where:

- S_{oe} = initial equilibrium bed slope [dimensionless]
- m = loading ratio (calculated as the ratio of the total sediment supply rate divided by the sediment transport rate of the initial equilibrium state) [dimensionless]
- ν = kinematic viscosity of the fluid [L^2/T]
- s = ratio of sediment density (ρ_s) to water density (ρ) [dimensionless]
- d = sediment size [L]

Chang and Yen (2002) assumed that the effect of the sedimentation parameter, $\nu^2/[(s - 1)gd^3]$, was negligible and, since the range of d/h during their experiments did not change much, both of those terms were eliminated from their analysis. They also performed nine flume experiments with two uniform fine gravel particle sizes (2.95 mm and 4.36 mm) and used a least mean square error regression analysis to fit the data, computing a dispersion coefficient equation:

$$\frac{K}{u_* d} = 10^{6.94} S_{oe}^{2.81} m^{-0.87} \quad (8)$$

For the range of experimental conditions tested within the Chang and Yen (2002) study, their equation yielded a wide range of K values, from 7.8 cm²/s to 113 cm²/s.

A series of papers by Furbish et al. (2012 a, b, and c) and Roseberry et al. (2012) describe the theory and details of bed-load sediment flux. Furbish et al. (2012c) use the description of particle velocity autocovariance (Taylor 1921) to calculate longitudinal diffusivity values in the range of 0.3 to 0.8 cm²/s. Although it was intended as a conceptual demonstration, as opposed to a process for calculating a specific value, one section of Furbish et al. (2012b) back calculated a dispersion coefficient of $K = 3.4$ cm²/s within the Fokker-Planck formulation. However, this result depends on simulation of specific assumptions about the travel distances for individual particle trajectories and a certain impact on downstream particle activity, which are uncertain and system specific parameters. Furbish et al. (2012b) also explain how the mean velocity and diffusivity are fundamentally related to each other, and how each are connected to the particle activity within the bed.

Table 2 and Table 3 summarize available relationships and data, respectively, for the bed-load dispersion coefficient. For the specific case of slope-dependent transport, Furbish et al. (2009) developed a relationship for bed-load dispersion based on a number of parameters as shown in Table 2. Interestingly, one of the key fundamental length scales was the active soil thickness, h_{AS} . Even more simple than the Elder (1959) equation (Eq. 2), Fan et al. (2014) used a diffusivity parameter that was proportional to the friction velocity, u_* , although it has different units ($L^2 T^{-3}$) and is not exactly described as a dispersion coefficient. Heyman et al. (2015) also found that streamwise dispersion scaled linearly with shear velocity ($K \approx 3.3d_{50}u_*$). Table 3 lists the available values for the bed-load dispersion coefficient found in the literature.

Table 2. Bed-load dispersion coefficient formulas and relations available in literature.

Source	Formula or relation for K^*	Notes/limitations
Furbish et al. 2009	$K = \alpha h_{AS} \left(\frac{d}{2} \right) N_a \left(1 - \frac{c}{c_m} \right)^2 \cos^2 \theta$	Diffusion-like coefficient in slope-dependent transport
Chang and Yen (2002)	$\frac{K}{u_* d} = 10^{6.94} S_{oe}^{2.81} m^{-0.87}$	Based on dimensional analysis
Fan et al. (2014)	Diffusivity $\propto u_*$	note: this diffusivity has different units
Heyman et al. (2015)	$K \approx 3.3d_{50}u_*$	Based on steep, shallow flume with uniform gravel

*Note: See Appendix for symbol definitions

Table 3. Bed-load dispersion coefficient values available in literature.

Source	Formula or value for K^*	Notes/limitations
Furbish et al. (2012b)	$K = 3.4 \text{ cm}^2/\text{s}$	Developed using a Fokker-Planck formulation
Furbish et al. (2012c)	$K = 0.3 \text{ to } 0.8 \text{ cm}^2/\text{s}$	Based on Taylor velocity autocorrelation
Drake et al. (1988)	$K_x = 4.6 \text{ cm}^2/\text{s}$ $K_y = 0.26 \text{ cm}^2/\text{s}$	Based on field observations, using MSD
Seizilles et al. (2014)	$K_y \approx 0.03 V_s d$	V_s is Stoke's settling velocity
Fan et al. (2014)	Diffusivity $\propto u_*$	note: this diffusivity has different units

*Note: See Appendix for symbol definitions

Most USACE developed transport models include computational dispersion capabilities; including AdH SEDLIB (Brown et al. 2014), the Coastal Modeling System (CMS) (Demirbilek and Rosati 2011; Lin et al. 2011; Sanchez et al. 2011a, and b), and HEC-RAS (USACE 2016; Gibson and Boyd 2014). Even though the hydraulic routing elements in the hydrologic sediment yield model in the HEC-Hydrologic Modeling System (HMS) included the (Fischer et al. 1979) analytical dispersion equation in developmental versions, HEC-HMS did not release that version due to difficulty parameterizing the coefficient. However, according to ERDC scientists, only AdH includes a bed-load specific approach, which adds a user-specified scaling coefficient to Equation 2. Further research is needed in determining the validity of the expression and the appropriate scaling coefficient for different applications.

3.1.2 Evidence of anomalous dispersion

Because of the simple analogy to well-established theory, bed-load dispersion and diffusion are usually discussed as representing normal (Fickian) behavior. However, growing literature questions this analogy, hypothesizing that bed-load is transported according to principles of anomalous (non-Fickian, non-normal) dispersion. Sun et al. (2015) flume data analysis showed that the classical ADE is incapable of explaining the bed-load sediment dispersion.

Normal dispersion affects the particle displacement variance proportionally with time (Eq. 5). However, anomalous dispersion produces particle spreading rates either larger (super-dispersion) or smaller (sub-dispersion) than the normal model. Mathematically, the general form of relating variance, in the x -direction, to time, t , is:

$$V(x) \propto t^{2\gamma} \quad (9)$$

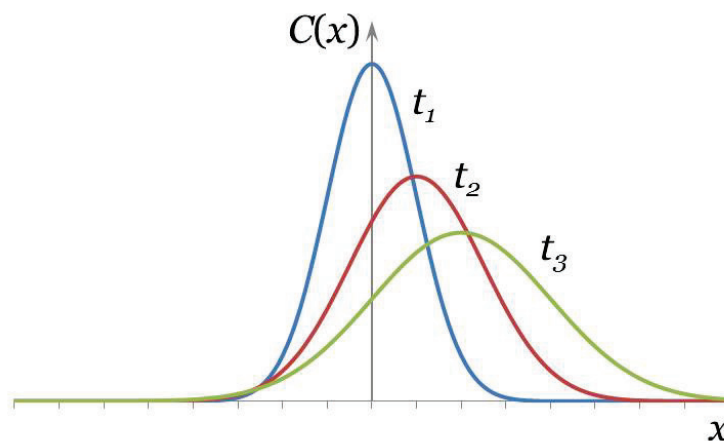
Where:

γ = scaling exponent for dispersion

The exponent, γ , is 0.5 for normal dispersion, $0 \leq \gamma < 0.5$ for sub-dispersion, or $0.5 < \gamma$ for super-dispersion.

In addition to a different relationship for the variance of the particle displacements, anomalous dispersion can also affect the shape of the probability density function. Normal dispersion typically approximates a Gaussian distribution, as shown in Figure 2. With this assumption, the peak drops and probabilities widen as time progresses, but the particle distribution remains symmetrical around the mean. However, it is possible for other types of distributions that are not symmetrical to describe the dispersion process. Indeed, Grigg (1969) and Yang and Sayre (1971) found that the stochastic behavior of sediment particles behaved according to a skewed gamma distribution, more like Figure 3, as opposed to a shape similar to those in Figure 2.

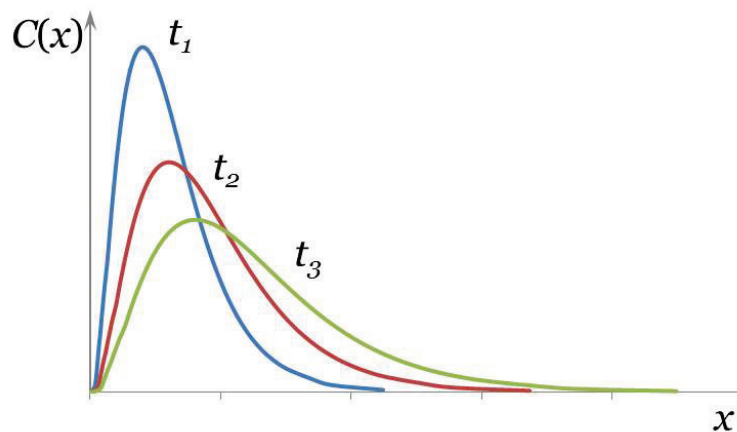
Figure 2. Example of dispersion that follows a Gaussian distribution, while also undergoing advection from left to right (progressing in time from $t_1 \rightarrow t_2 \rightarrow t_3$).



Even during solute transport, anomalous dispersion occurs initially. According to Fischer et al. (1979), normal dispersion does not apply at normalized distances ($x' = x\varepsilon_t/(\bar{u}W^2)$) less than 0.4, where ε_t is a transverse mixing coefficient. In the initial spreading phase, the distribution can skew, pushing the peak left of the mean and elongating the leading edge (i.e., a heavy tail) (Hill et al. 2010), as sketched in Figure 3. Liebault et al. (2012), claims that frontrunners, or particles at the leading edge, can travel very quickly, resulting in the heavy-leading-tail distribution shape. Bed-load transport literature refers to several types of heavy-tailed distributions such as lognormal, inverse gamma, exponential, power-law, Weibull, Pareto, and Poisson (Einstein 1972; Hassan et al. 1991; Hill et al. 2010; Haschenburger 2013). Hubbell and Sayre's (1964) data also showed

a heavy-leading-tail distribution. Identifying which distribution(s) best describe bed-load dispersion is an active area of research and an important consideration when applying dispersion. Fischer et al. (1979) acknowledged that the longitudinal concentration distribution of solutes only becomes normal asymptotically, and for normalized distances greater than $x' = 1.0$ it is a sufficient approximation. However, the work of Phillips et al. (2013) suggested that the asymptotic limit of bed-load dispersion is super-dispersive. Fischer et al. (1979) acknowledged two examples of dispersion studies that reportedly never showed normal, Fickian behavior (Nordin and Sabol 1974, and Day 1975).

Figure 3. Example of dispersion that produces a heavy-tail distribution (progressing in time from $t_1 \rightarrow t_2 \rightarrow t_3$).



To investigate normal vs. anomalous dispersion, Hassan et al. (2013) studied 64 gravel-bed field tracer experiments to determine whether the tracers exhibited heavy-tail or thin-tail distributions. Thin-tail distributions indicate normal dispersive behavior, while heavy-tail distributions indicate anomalous dispersion. Of the 64 surveys, 51 exhibited thin-tail distributions and 8 more could have been considered thin-tail based on the definition of the ‘tail’. Liebault et al. (2012) performed experimental work on the Bouinenc Torrent, a tributary to the Bléone River in southeast France. Their work classified two out of three Bouinenc Torrent experimental cases as heavy-tail distributions, but upon reanalysis, Hassan et al. (2013) classified all three Bouinenc Torrent surveys as thin-tail distributions. Therefore, the work of Hassan et al. (2013) indicates normal dispersive behavior in most cases, but much of the literature does not agree.

Recent non-Fickian literature (e.g., Schumer et al. 2009; Bradley et al. 2010; Martin et al. 2012) points to the work of Nikora et al. (2001 and

2002) as a key concept-framing work for anomalous bed-load dispersion. Nikora et al. (2001) identified three different scales of bed-load transport; namely the local, intermediate, and global scales. The local scale is an individual particle's trajectory within one saltating movement. The particle trajectory between two successive periods of rest is the intermediate scale. The third scale, combining multiple intermediate scales, is the global scale. Nikora et al. (2002) suggest that each range can exhibit a different type of dispersive behavior, i.e., each can have a different exponent, γ , in Equation 9. Bialik et al. (2015) estimates the transition points between scales.

3.1.2.1 Local scale

The dispersive spreading within the local range is caused by the variations of particle velocities, accelerations, hop distances, and travel times within a single jump (Fathel et al. 2015). There are a number of different types of distributions used to characterize the velocities and/or travel times at the local scale, such as an exponential distribution (Roseberry et al. 2012, Furbish et al. 2012b, and Lajeunesse et al. 2010) or a truncated Gaussian distribution (Martin et al. 2012, Ancely and Heyman 2014). The local scale is generally considered super-dispersive (Khzeri and Chanson 2015, Bialik et al. 2015). Martin et al. (2012) studied the physical mechanisms at the local scale, observing super-dispersion with an approximate exponent, γ , of 0.8 at times from 0.2 to 1 sec and then $\gamma = 0.7$ beyond 1 sec in their experiments. These exponent values were in good agreement with the super-dispersive observations of Nikora et al. (2002), where γ was 0.77 to 0.87 for the Balmoral Irrigation Canal in North Canterbury, New Zealand. Martin et al. (2012) hypothesized that the super-dispersive behavior at the local scale is caused by particle inertia and fluid drag, along with possible influences from hydrodynamic and boundary effects.

Huang et al. (2011) studied the displacement of particles at the molecule-scale and observed the full transition from pure ballistic motion ($\gamma = 1$) to diffusive Brownian motion as the movement became more defined by the fluid behavior. However, a growing amount of literature points to super-ballistic ($\gamma > 1$) motion for bed sediment dispersion (Campagnol et al. 2015, Olinde and Johnson 2015, Phillips et al. 2013, Martin et al. 2012 and 2014). Campagnol et al. (2015) showed that if the particles are exhibiting short saltating hops (i.e., weak transport conditions), the super-ballistic behavior at the local scale is from the strong acceleration experienced by the particles.

3.1.2.2 *Intermediate scale*

The intermediate scale is generally characterized by the particle wait times between saltating movements. In the words of H. Einstein (1942): “bed-load movement is to be considered as the motion of bed particles in quick steps with comparatively long intermediate periods of rest” (Nikora et al. 2001). Particle wait times were significant enough in the work of Bennett and Nordin (1977) that the researchers considered dispersion of suspended sediment negligible with respect to dispersion caused by storage of particles in the streambed.

Literature offers conflicting explanations for dispersive behavior in the intermediate scale. Nikora et al. (2002) stated that the intermediate scale could exhibit normal, super-, or sub-dispersive behavior. The researchers’ analysis of Balmoral Canal (North Canterbury, New Zealand) showed super-dispersion in the intermediate scale, but Zhang et al. (2012) later reclassified this work as the local scale. Martin et al. (2012) explained challenges with identifying the transition between local and intermediate scales, but stated that the heavy-tailed wait times cause the behavior to become sub-dispersive beyond the local scale. Zhang et al. (2012) also found sub-dispersive behavior in the intermediate range. Zhang et al. (2012) point out that experimental observations can sometimes be challenging since newly introduced tracer particles tend to be more mobile than average, causing super-dispersion to be overestimated, while on the other hand, the transport of the most mobile particles can be limited by the window of observation, which could cause super-dispersion to be underestimated. Perhaps this helps explain some of the conflicting evidence found in the literature.

It is important to note that the vertical exchange within the bed has an important influence on the particle storage and release, or the particle wait time between jumps (Parker et al. 2000; Pelosi et al. 2016). Dispersion at intermediate and global scales depends on this vertical storage and exchange. Therefore, a sediment model could completely ignore the dispersion term of the ADE, but if it simulates vertical exchange behavior, the model results will still exhibit a dispersive effect (Ferguson and Hoey 2002; Ganti et al. 2010).

3.1.2.3 Global scale

The global scale is the largest of the three spatial scales, and it is defined as the combination of many intermediate scales. The sub-, normal, or super-dispersion of the global range is even more controversial than agreement at the intermediate scale. Meerschaert et al. (2008) proposed a tempered model to capture the slow transition from sub-dispersion to a normal dispersion limit. Voepel et al. (2013) studied waiting times and the transition between the intermediate and global scales, reporting transitions on the order of seconds-to-minutes in flume studies, while it may take days to years in natural channels. Drake et al. (1988), assuming normal dispersion, was the first to analyze data from Duck Creek near Pinedale, Wyoming. Then, Nikora et al. (2002) reanalyzed the data to find sub-dispersive behavior and classified it in the global range. Zhang et al. (2012) advanced Meerschaert et al.'s (2008) tempered stable model and reanalyzed the Duck Creek data further. Zhang et al. (2012) agreed with Nikora et al. (2002) that it was sub-dispersive, but re-classified the range as intermediate instead of global. Bialik et al. (2012) confirmed the existence of at least three ranges of scales and identified dispersive exponents that decreased from one range to the next. However, Zhang et al. (2012) might classify all of Bialik et al.'s (2012) observations within the local and intermediate ranges.

Sayre and Hubbell (1965) performed field experiments using 40 pounds of radioactive tracers in the North Loup River, Nebraska. Their observations revealed an episodic behavior in which tracer particles were buried under migrating dunes and then released by scour. Assuming exponential distributions for step distance and rest periods, Sayre and Hubbell (1965), in line with Einstein (1972), derived concentration profiles. However, the observed plumes had heavier tails than the Sayre and Hubbell model indicated, and Bradley et al. (2010) and Zhang et al. (2012) proposed new models in an attempt to simulate the Sayre and Hubbell (1965) observations.

Olinde and Johnson (2015) used radio frequency identification and accelerometer embedded tracers to show that for mountain streams, the variance can actually begin to decrease after some amount of time due to spatial changes in grain sizes and reach morphology. Olinde and Johnson (2015) found that the displacements and step lengths exhibited thin-tailed distributions, while rest times in mountainous, snow-impacted streams were described by heavy-tailed power law distributions resulting in an

overall super-dispersive behavior of the tracers. Pelosi et al. (2016) showed that the 2D EBME model, which resembled field data well, also exhibited super-dispersive behavior. Therefore, there is literature evidence for sub-, normal, and super-dispersion at the global scale.

3.1.3 Fractional Advection–Dispersion Equation (ADE)

Recent bed-load dispersion modeling literature investigates the difference between anomalous dispersion and the classic ADE. Specifically, the dispersion term of the ADE for anomalous dispersive behavior has a fractional derivative, yielding the fractional advection dispersion equation (FADE):

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = K \frac{\partial^{(1/\gamma)} C}{\partial x^{(1/\gamma)}} \quad (10)$$

where γ is the scaling exponent, the dispersion coefficient K has dimensions of $[L^{(1/\gamma)}/T]$, and u is the mean bed-load material velocity. As before, $\gamma = 0.5$ yields the normal, or Fickian, dispersive behavior and the classic ADE. The work of Sabzikar et al. (2015) applied tempered fractional calculus to describe the complexities of the mathematical behavior. Benson et al. (2013) connected the FADE to random walks governed by heavy-tailed probability distributions and discussed several Eulerian and Lagrangian numerical methods for treating the fractional partial derivative.

Ganti et al. (2010) demonstrated how a formulation of the Exner equation for sediment conservation, which incorporates a probability distribution of step lengths, having heavy-tails in real rivers, leads to an anomalous advection-dispersion formulation. Fan et al. (2014) developed a Langevin formulation of bed-load modeling using a balance of the forces on particles. Pelosi and Parker (2014) offered an informative comparison between numerical frameworks based on flux and those based on entrainment. Pelosi et al. (2014 and 2016) used the Parker et al. (2000) framework of the Exner equation to simulate complexities within the bed, departing from the ‘active layer’ concept and producing vertical dispersion.

Sun et al. (2015) also showed that classical ADE was incapable of explaining the behavior of laboratory flume dispersion, and the researchers’ analysis used fractional derivatives for the time derivative (left-most term of Eq. 10) as well as the spatial derivative (right-most term of Eq. 10). Once

implemented, the Sun et al. (2014) FADE model was relatively insensitive to the dispersion coefficient chosen; the tested range covered two orders of magnitude. Zhang et al. (2014) developed the use of a simpler (than FADE) approach called the subordinated advection equation (SAE) to explain various historical data sets for sediment dispersion across a wide range of scales. Numerous other recent numerical models targeted stochastic, anomalous dispersion of bed-load transport including: Schumer et al. (2009), Bradley et al. (2010), Lajeunesse et al. (2010), Zhang et al. (2012), Lajeunesse et al. (2013), Martin et al. (2014), Ancey and Heyman (2014), and others.

3.2 Transverse dispersion

If the dispersion behavior is the same in the longitudinal (i.e., streamwise) and transverse (i.e., cross-stream) directions, it is considered isotropic; otherwise it is anisotropic. The majority of the recent literature considers it anisotropic (Furbish et al. 2012c; Nikora et al. 2001; Roseberry et al. 2012; Seizilles et al. 2014; etc.). Samson et al. (1998) studied the motion of spheres down a sloped bumpy plane and found that the dispersion could be described by anisotropic normal dispersion. Samson et al. (1998) found that the longitudinal dispersion coefficient had a constant value independent of the sphere size and slope, while the transverse dispersion coefficient was found to decrease with the plane slope and scale with the size of the spheres. Interestingly, Samson et al. (1998) found that the longitudinal dispersion coefficient observed was smaller than the transverse coefficient, a finding contradicted by the results of Roseberry et al. (2012) where the longitudinal dispersion coefficient was found to be larger.

Seizilles et al. (2014) also found that cross-stream dispersion was entirely consistent with normal Fickian behavior, in contrast to the observations of Nikora et al. (2002). Bialik et al. (2012) found only two scales of dispersion in the transverse direction, compared to at least three in the longitudinal direction. Similar to Samson et al. (1998), Bialik et al. (2012) stated that the spread of particles in the longitudinal direction mostly depended on turbulence, instead of relative particle size, while spreading in the transverse direction did depend on relative particle size. Expanding upon that, Barati et al. (2015) used numerical modeling to show that the main mechanism for longitudinal dispersion is the flow condition, while the main mechanisms for transverse dispersion are collisions, both inter-particle and particle-bed.

4 Conclusion

Many investigators have successfully applied Fickian models to their data, but growing evidence suggests that non-Fickian, anomalous, anisotropic processes offer value in representing bed-load dispersion. Some researchers agree that dispersion can be characterized by three different scales; local, intermediate, and global. A majority agree that the local scale is characterized by super-dispersive behavior. There are numerous findings, even in just the last couple of years, which present significant disagreements about the global scale, whether it exhibits super-, sub-, or normal dispersion. As witnessed by the number of recent publications, sediment bed-load dispersion is an active area of research, and the published values for the bed-load dispersion coefficient vary greatly. Some formulas and recommended values for the dispersion coefficient are shown in Table 2 and Table 3. For numerical modeling, numerical dispersion coefficients are relative to the context of all considerations within a modeling framework and may not be directly comparable across studies. Future work includes the evaluation and improvement of USACE sediment transport models using the laboratory experiments of Gibson et al. (2016) to test the dispersion frameworks discussed herein.

References

- Abraham, D., R. Kuhnle, and A. Odgaard. 2011. Validation of bed-load transport measurements with time-sequenced bathymetric data. *Journal of Hydraulic Engineering* 137(7): 723–728.
- Ancey, C., and J. Heyman. 2014. A microstructural approach to bed load transport: Mean behaviour and fluctuations of particle transport rates. *Journal of Fluid Mechanics* 744: 129–168.
- Bagnold, R. A. 1956. The flow of cohesionless grains in fluids. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 249(964): 235–297.
- Barati, R., S. A. A. S. Neyshabouri, and G. Ahmadi. 2015. Study of sediment transport under saltation regime in rough bed: Bed-load grain diffusion. In *Proceedings, 36th IAHR World Congress. 28 June–3 July, The Hague, the Netherlands*.
- Bennett, J. P., and C. F. Nordin. 1977. Simulation of sediment transport and armoring. *Hydrological Sciences Journal* 22(4): 555–569.
- Benson, D. A., M. M. Meerschaert, and J. Revielle. 2013. Fractional calculus in hydrodynamic modeling: A numerical perspective. *Advances in Water Resources* 51: 479–497.
- Bialik, R. J., V. I. Nikora, and P. M. Rowiński. 2012. 3D lagrangian modelling of saltating particles diffusion in turbulent water flow. *Acta Geophysica* 60(6): 1639–1660.
- Bialik, R. J., V.I. Nikora, M. Karpiński, and P. M. Rowiński. 2015. Diffusion of bedload particles in open-channel flows: Distribution of travel times and second-order statistics of particle trajectories. *Environmental Fluid Mechanics* 15(6): 1281–1292.
- Bradley, D. N., G. E. Tucker, and D. A. Benson. 2010. Fractional dispersion in a sand bed river. *Journal of Geophysical Research: Earth Surface* 115(F1).
- Brown, G. L., C. Trahan, J. N. Tate, and G. Savant. 2014. *SEDLIB multiple grain sized mixed sediment library: Technical manual. v1.2*. Vicksburg, MS: U.S. Army Engineer Research and Development Center.
- Brown, R. 1828. XXVII. A brief account of microscopical observations made in the months of June, July and August 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *The Philosophical Magazine, Or Annals of Chemistry, Mathematics, Astronomy, Natural History and General Science Series 2* 4(21): 161–173.
- Campagnol, J., A. Radice, F. Ballio, and V. Nikora. 2015. Particle motion and diffusion at weak bed load: accounting for unsteadiness effects of entrainment and disentrainment. *Journal of Hydraulic Research* 53(5): 633–648.

- Carling, P. A., H. Orr, and A. Kelsey. 2006. The dispersion of magnetite bedload tracer across a gravel point-bar and the development of heavy-mineral placers. *Ore Geology Reviews* 28(4): 402–416.
- Chang, S., and C. Yen. 2002. Simulation of bed-load dispersion process. *Journal of Hydraulic Engineering* 128(3): 331–342.
- Day, T. J. 1975. Longitudinal dispersion in natural channels. *Water Resources Research* 11(6): 909–918.
- Demirbilek, Z., and J. D. Rosati. 2011. *Verification and validation of the Coastal Modeling System: Report 1, Summary Report*. ERDC/CHL TR-11-10. Vicksburg, Miss: U.S. Army Engineer Research and Development Center.
- Deng, Z., L. Bengtsson, V. Singh, and D. Adrian. 2002. Longitudinal dispersion coefficient in single-channel streams. *Journal of Hydraulic Engineering* 128(10): 901–916.
- Disley, T., B. Gharabaghi, A. Mahboubi, and E. McBean. 2015. Predictive equation for longitudinal dispersion coefficient. *Hydrological Processes* 29(2): 161–172.
- Drake, T. G., R. L. Shreve, W. E. Dietrich, P. J. Whiting, and L. B. Leopold. 1988. Bedload transport of fine gravel observed by motion-picture photography. *Journal of Fluid Mechanics* 192: 193–217.
- Du Boys, M. 1879. The rhone and streams with movable beds. *Annals Des Pontes Et Chaussees* 18: 22.
- Einstein, A. 1905. On the movement of small particles suspended in stationary liquids required by the molecular-kinetic theory of heat. *Annalen Der Physik* 17: 549–560.
- Einstein, H. 1972. Bedload transport as a probability problem. In *Sedimentation: Symposium to honor H. A. Einstein*, ed. H. W. Shen, 105–108. Fort Collins, CO: Colorado State University.
- Einstein, H. A. 1942. Formulas for the transportation of bed load. *Transactions of the American Society of Civil Engineers* 107(Paper 2140): 561–597.
- Elder, J. 1959. The dispersion of marked fluid in turbulent shear flow. *Journal of Fluid Mechanics* 5(04): 544–560.
- Fan, N., D. Zhong, B. Wu, E. Foufoula-Georgiou, and M. Guala. 2014. A mechanistic-stochastic formulation of bed load particle motions: From individual particle forces to the Fokker-Planck equation under low transport rates. *Journal of Geophysical Research: Earth Surface* 119(3), 464–482.
- Fathel, S. L., D. J. Furbish, and M. W. Schmeeckle. 2015. Experimental evidence of statistical ensemble behavior in bed load sediment transport. *Journal of Geophysical Research: Earth Surface* 120(11), 2298–2317.
- Ferguson, R. I., and T. B. Hoey. 2002. Long-term slowdown of river tracer pebbles: Generic models and implications for interpreting short-term tracer studies. *Water Resources Research* 38(8): 17-1–17-11.

- Fick, A. 1855. V. On liquid diffusion. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 10(63): 30–39.
- Fischer, H., E. List, R. Koh, J. Imberger, and N. Brooks. 1979. *Mixing in inland and coastal waters*. San Diego, CA: Academic Press.
- Furbish, D. J., P. K. Haff, W. E. Dietrich, and A. M. Heimsath. 2009. Statistical description of slope-dependent soil transport and the diffusion-like coefficient. *Journal of Geophysical Research: Earth Surface* 114(F3).
- Furbish, D. J., P. K. Haff, J. C. Roseberry, and M. W. Schmeeckle. 2012a. A probabilistic description of the bed load sediment flux: 1. theory. *Journal of Geophysical Research* 117(F3).
- Furbish, D. J., J. C. Roseberry, and M. W. Schmeeckle. 2012b. A probabilistic description of the bed load sediment flux: 3. The particle velocity distribution and the diffusive flux. *Journal of Geophysical Research* 117(F3).
- Furbish, D. J., A. E. Ball, and M. W. Schmeeckle. 2012c. A probabilistic description of the bed load sediment flux: 4. Fickian diffusion at low transport rates. *Journal of Geophysical Research* 117(F3).
- Ganti, V., M. Meerschaert, E. Foufoula-Georgiou, E. Viparelli, and G. Parker. 2010. Normal and anomalous dispersion of gravel tracers in rivers. *Journal of Geophysical Research* 115: 2156–2202.
- Gibson, S. 2009. The temporal and gradational trends of sand infiltration in a gravel bed. PhD dissertation, University of California, Davis.
- Gibson, S.A., C. Nygaard, and P. Scafani. 2010. Mobile bed modeling of the Cowlitz River using HEC-RAS: Assessing flood risk and impact due to system sediment. In *Proceedings, 2nd Joint Federal Interagency Conference, 27 June–1 July*, Las Vegas, NV.
- Gibson, S., R. Heath, D. Abraham, and D. Schoellhamer. 2011. Visualization and analysis of temporal trends of sand infiltration into a gravel bed. *Water Resources Research* 47(12).
- Gibson, S.A., and P. Boyd. 2014. Modeling long term alternative for sustainable sediment management using operational sediment transport rules. In *Reservoir Sedimentation*, eds. A. J. Schleiss, G. De Cesare, M. J. Franca, and M. Pfister, 229–236. Boca Raton, FL: CRC Press.
- Gibson, S., R. Heath, K. Ramos, J. Sharp, D. Abraham, and L. Kavaas. 2016. Experimental determination of bedload dispersion coefficients. Unpublished manuscript.
- Grigg, N. S. 1969. Motion of single particles in sand channels. PhD diss., Colorado State University.
- Haschenburger, J. 2013. Tracing river gravels: Insights into dispersion from a long-term field experiment. *Geomorphology* 200: 121–131.

- Hassan, M. A., M. Church, and A. P. Schick. 1991. Distance of movement of coarse particles in gravel bed streams. *Water Resources Research* 27(4): 503–511.
- Hassan, M. A., H. Voepel., R. Schumer, G. Parker, and L. Fraccarollo. 2013. Displacement characteristics of coarse fluvial bed sediment. *Journal of Geophysical Research: Earth Surface* 118(1): 155–165.
- Heyman, J., P. Bohorquez, and C. Ancey. 2015. Exploring the physics of sediment transport in non-uniform super-critical flows using a large dataset of particle trajectories. No. EPFL-ARTICLE-210219. Hoboken, NJ: Wiley-Blackwell publishing.
- Hill, K. M., L. DellAngelo, and M. M. Meerschaert. 2010. Heavy-tailed travel distance in gravel bed transport: An exploratory enquiry. *Journal of Geophysical Research* 115(F2).
- Holley, E. R. 1969. Unified view of diffusion and dispersion. *Journal of the Hydraulics Division* 95(2): 621–632.
- Huang, R., I. Chavez, K. M. Taute, B. Lukić, S. Jeney, M. G. Raizen, and E. Florin. 2011. Direct observation of the full transition from ballistic to diffusive brownian motion in a liquid. *Nature Physics* 7(7): 576–580.
- Hubbell, D. W., and W. W. Sayre. 1964. Sand transport studies with radioactive tracers. *Journal of the Hydraulics Division* 90(3): 39–68.
- Iwasa, Y., and S. Aya. 1991. Predicting longitudinal dispersion coefficient in open channel flows. In *Proceedings of International Symposium on Environmental Hydraulics*, 16–18 Dec, Hong Kong, China, eds. J. H. W. Lee and Y. K. Cheung, 505–510.
- Kashefipour, S. M., and R. A. Falconer. 2002. Longitudinal dispersion coefficients in natural channels. *Water Research* 36(6): 1596–1608.
- Kennedy, V. C. and D. L. Kouba. 1970. Fluorescent sand as a tracer of fluvial sediment. Professional Paper 562E. Washington, DC: United States Department of the Interior, U.S. Geological Survey.
- Khzeri, N., and H. Chanson. 2015. Turbulent velocity, sediment motion and particle trajectories under breaking tidal bores: simultaneous physical measurements. *Environmental Fluid Mechanics* 15(3): 633–650.
- Koussis, A. D., and J. Rodríguez-Mirasol. 1998. Hydraulic estimation of dispersion coefficient for streams. *Journal of Hydraulic Engineering* 124(3): 317–320.
- Lajeunesse, E., L. Malverti, and F. Charru. 2010. Bed load transport in turbulent flow at the grain scale: Experiments and modeling. *Journal of Geophysical Research: Earth Surface* 115(F4).
- Lajeunesse, E., O. Devauchelle, M. Houssais, and G. Seizilles. 2013. Tracer dispersion in bedload transport. *Advances in Geosciences* 37: 1-6.

- Liébault, F., H. Bellot, M. Chapuis, S. Klotz, and M. Deschâtres. 2012. Bedload tracing in a high-sediment-load mountain stream. *Earth Surface Processes and Landforms* 37(4): 385–399.
- Lin, L., Z. Demirbilek, R. Thomas, and J. Rosati. 2011. Verification and validation of the Coastal Modeling System: Report 2. ERDC/CHL TR-11-10. Vicksburg, Miss: U.S. Army Engineer Research and Development Center.
- Liu, H. 1977. Predicting dispersion coefficient of streams. *Journal of the Environmental Engineering Division* 103(1): 59–69.
- Martin, R. L., D. J. Jerolmack, and R. Schumer. 2012. The physical basis for anomalous diffusion in bed load transport. *Journal of Geophysical Research: Earth Surface* 117(F1).
- Martin, R. L., P. K. Purohit, and D. J. Jerolmack. 2014. Sedimentary bed evolution as a mean-reverting random walk: Implications for tracer statistics. *Geophysical Research Letters* 41(17): 6152–6159.
- McQuivey, R. S., and T. N. Keefer. 1974. Simple method for predicting dispersion in streams. *Journal of the Environmental Engineering Division* 100(4): 997–1011.
- Meerschaert, M. M., Y. Zhang, and B. Baeumer. 2008. Tempered anomalous diffusion in heterogeneous systems. *Geophysical Research Letters* 35(17).
- Meyer-Peter, E., and R. Müller. 1948. Formulas for bed-load transport. In *Proceedings, International Association of Hydraulic Research 2nd Congress*, 7–9 June, Stockholm, Sweden. 39–64.
- Nikora, V., J. Heald, D. Goring, and I. McEwan. 2001. Diffusion of saltating particles in unidirectional water flow over a rough granular bed. *Journal of Physics A: Mathematical and General* 34(50): L743.
- Nikora, V., H. Habersack, T. Huber, and I. McEwan. 2002. On bed particle diffusion in gravel bed flows under weak bed load transport. *Water Resources Research* 38(6): 17-1–17-9.
- Nordin, C. F., and G. V. Sabol. 1974. Empirical data on longitudinal dispersion in rivers. Water Resources Investigation 74-20. Washington, D.C.: United States Department of the Interior, U.S. Geological Survey.
- Olinde, L., and J. P. Johnson. 2015. Using RFID and accelerometer-embedded tracers to measure probabilities of bed load transport, step lengths, and rest times in a mountain stream. *Water Resources Research* 51(9): 7572–7589.
- Parker, G., C. Paola, and S. Leclair. 2000. Probabilistic exner sediment continuity equation for mixtures with no active layer. *Journal of Hydraulic Engineering* 126(11): 818–826.
- Pearson, K. 1905. The problem of the random walk. *Nature* 72(1865): 294.

- Pelosi, A., R. Schumer, G. Parker, and R. I. Ferguson. 2016. The cause of advective slowdown of tracer pebbles in rivers: Implementation of exner-based master equation for coevolving streamwise and vertical dispersion. *Journal of Geophysical Research: Earth Surface* 121(3).
- Pelosi, A., and G. Parker. 2014. Morphodynamics of river bed variation with variable bedload step length. *Earth Surface Dynamics* 2(1): 243–253.
- Pelosi, A., G. Parker, R. Schumer, and H. Ma. 2014. Exner-Based master equation for transport and dispersion of river pebble tracers: Derivation, asymptotic forms, and quantification of nonlocal vertical dispersion. *Journal of Geophysical Research: Earth Surface* 119(9): 1818–1832.
- Phillips, C. B., R. L. Martin, and D. J. Jerolmack. 2013. Impulse framework for unsteady flows reveals superdiffusive bed load transport. *Geophysical Research Letters* 40(7): 1328–1333.
- Roseberry, J. C., M. W. Schmeeckle, and D. J. Furbish. 2012. A probabilistic description of the bed load sediment flux: 2. particle activity and motions. *Journal of Geophysical Research* 117(F3).
- Sabzikar, F., M. M. Meerschaert, and J. Chen. 2015. Tempered fractional calculus. *Journal of Computational Physics* 293: 14–28.
- Sahay, R., and S. Dutta. 2009. Prediction of longitudinal dispersion coefficients in natural rivers using genetic algorithm. *Hydrology Research* 40(6): 544–552.
- Samson, L., I. Ippolito, G. G. Batrouni, and J. Lemaitre. 1998. Diffusive properties of motion on a bumpy plane. *The European Physical Journal B—Condensed Matter and Complex Systems* 3(3): 377–385.
- Sánchez, A., and W. Wu 2011. A non-equilibrium sediment transport model for coastal inlets and navigation channels. *Journal of Coastal Research Special Issue* (59): 39–48.
- Sanchez, A., W. Wu, T. M. Beck, H. Li, and J. Rosati. 2011a. *Verification and validation of the Coastal Modeling System: Report 3*. ERDC/CHL TR-11-10 rept. 3. Vicksburg, Miss: U.S. Engineer Research and Development Center.
- Sanchez, A., W. Wu, T. M. Beck, H. Li, J. D. Rosati, Z. Demirbilek, and M. E. Brown. 2011b. *Verification and validation of the Coastal Modeling System: Report 4*. ERDC/CHL TR-11-10 rept. 4. Vicksburg, Miss: Engineer Research and Development Center.
- Sayre, W. W., and D. W. Hubbell. 1965. Transport and dispersion of labeled bed material, North Loup River, Nebraska. Professional Paper 433-C. Washington, DC: U.S. Department of the Interior, U.S. Geological Survey.
- Schumer, R., M. M. Meerschaert, and B. Baeumer. 2009. Fractional advection-dispersion equations for modeling transport at the earth surface. *Journal of Geophysical Research: Earth Surface* 114(F4)
- Seizilles, G., E. Lajeunesse, O. Devauchelle, and M. Bak. 2014. Cross-stream diffusion in bedload transport. *Physics of Fluids (1994–Present)* 26(1): 013302.

- Seo, I. W., and T. S. Cheong. 1998. Predicting longitudinal dispersion coefficient in natural streams. *Journal of Hydraulic Engineering* 124(1): 25–32.
- Sun, H., D. Chen, Y. Zhang, and L. Chen. 2015. Understanding partial bed-load transport: Experiments and stochastic model analysis. *Journal of Hydrology* 521: 196–204.
- Tate, J., C. Berger, and R. McAdory. 2009. Houston -Galveston Ship Channels sedimentation model validation. In *Proceedings, 33rd International Association of Hydraulic Engineering and Research (IAHR) Biennial Congress, 9–14 August, Vancouver, British Columbia*.
- Taylor, G. 1921. Diffusion by continuous movements. *Proceedings of the London Mathematical Society* 20: 196–211.
- Taylor, G. 1953. Dispersion of soluble matter in solvent flowing slowly through a tube. *Proceedings of the Royal Society of London A: Mathematical and Physical Sciences* 219(1137): 186–203.
- Taylor, G. 1954. The dispersion of matter in turbulent flow through a pipe. *Proceedings of the Royal Society of London A: Mathematical and Physical Sciences* 223(1155): 446–468.
- U.S. Army Corps of Engineers (USACE). 2016. HEC-RAS: River analysis system hydraulic reference manual. V. 5.0. CPD-69. Davis, CA: USACE Institute for Water Resources, Hydrologic Engineering Center.
- Voepel, H., R. Schumer, and M. A. Hassan. 2013. Sediment residence time distributions: Theory and application from bed elevation measurements. *Journal of Geophysical Research: Earth Surface* 118(4): 2557–2567.
- Wong, M., G. Parker, P. DeVries, T. M. Brown, and S. J. Burges. 2007. Experiments on dispersion of tracer stones under lower-regime plane-bed equilibrium bed load transport. *Water Resources Research* 43(3).
- Yalin, M. 1977. *Mechanics of sediment transport*, 2nd edition. Oxford, NY: Pergamon Press.
- Yang, C. T. and W. Sayre. 1971. Longitudinal Dispersion of bed-material particles. *Journal of the Hydraulics Division* 97(7): 907–921.
- Zhang, Y., M. M. Meerschaert, and A. I. Packman. 2012. Linking fluvial bed sediment transport across scales. *Geophysical Research Letters* 39(20).
- Zhang, Y., R. L. Martin, D. Chen, B. Baeumer, H. Sun, and L. Chen. 2014. A subordinated advection model for uniform bed load transport from local to regional scales. *Journal of Geophysical Research: Earth Surface* 119(12): 2711–2729.

Appendix: Definitions of Symbols

<i>Symbol</i>	<i>Description [units; M=mass, L = length, T=time]</i>
A	= Area of control volume normal to the flux [L^2]
C	= Concentration (sediment or solute) [typically M / L^3]
c	= Volumetric particle concentration [dimensionless]
c_m	= Natural consolidated (maximum) volumetric particle concentration (sediment or solute) [dimensionless]
d	= Sediment size [L]
d_{50}	= Median grain diameter [L]
D	= Diffusion coefficient [typically L^2 / T]
g	= gravitational acceleration [L / T^2]
h	= Fluid depth [L]
h_{AS}	= Active soil layer thickness [L]
K	= Dispersion coefficient [typically L^2 / T]
K_x	= Dispersion coefficient in the x -direction [typically L^2 / T]
K_y	= Dispersion coefficient in the y -direction [typically L^2 / T]
m	= Loading ratio of total sediment supply rate to the sediment transport rate of the initial equilibrium state [dimensionless]
MSD	= Mean Squared Displacement [L^2]
N_a	= Particle activation rate [T^{-1}]
q	= Solute mass flux [$M / (L^2 T)$]
S	= Slope of the channel [dimensionless]
s	= Ratio of sediment density to water density [dimensionless]
S_{oe}	= Initial equilibrium bed slope [dimensionless]
Δt	= Time step [T]
\bar{t}_1, \bar{t}_2	= Mean times of passage of tracer at locations x_1 and x_2 [T]
U	= Stream-wise velocity [L / T]
\bar{u}	= Mean stream-wise velocity [L / T]
u_*	= Friction velocity [L / T]
$V(x)$	= Variance of particle displacements in the x -direction [L^2]
V_s	= Stoke's settling velocity [L / T]
W	= Width of the channel [L]
x	= Particle displacement [L]
x'	= Normalized distance [dimensionless]
$\langle x \rangle$	= Mean particle displacement [L]
α	= Coefficient [dimensionless]
γ	= Scaling exponent for diffusive behavior [dimensionless]
γ_x, γ_y	= Diffusive exponents in the x - and y -directions [dimensionless]
ε_t	= Transverse mixing coefficient [L^2 / T]
θ	= Angle of land-surface slope [dimensionless]
κ	= Diffusivity [L^2 / T]
ν	= Kinematic viscosity of the fluid [L^2 / T]
ρ	= Fluid density [M / L^3]
ρ_s	= Sediment density [M / L^3]

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